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The group of self-homotopy classes of SO(4)☆

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Abstract

We calculate the groups of self-homotopy classes of $S^3 \times SO(3)$ and SO(4).

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1. Introduction

For a topological group G and a space X, let $[X, G]$ be the set of homotopy classes of maps from X into G. It inherits a group structure from G. This group has been studied by Whitehead [15] and others. Its structure is still unknown except for a few special cases. The group $[G, G]$ seems to be particularly interesting; we denote $[G, G]$ by $\mathscr{H}(G)$ in order to avoid confusion with the commutator subgroup of G. We have calculated $\mathscr{H}(G)$ for various simple Lie groups G of small rank [6,8–10]. The purpose of this note is to investigate the group $\mathscr{H}(G)$ in case G is a compact, connected, semi-simple, and non-simple Lie group of rank 2, that is, G is isomorphic to one of $S^3 \times S^3$, $S^3 \times SO(3)$, SO(4), and $SO(3) \times SO(3)$ (cf. Section 2). The group $\mathscr{H}(S^3 \times S^3)$ was given in [6] (cf. Proposition 2.1 below). In this paper we will determine $\mathscr{H}(S^3 \times SO(3))$, $\mathscr{H}(SO(4))$, and $\mathrm{nil}\,\mathscr{H}(SO(3) \times SO(3))$, where nil denotes the nilpotency class. Our calculations entail the following three results.

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Theorem 1.1.

G	$\mathrm{S}^3 \times \mathrm{S}^3$	$\mathrm{S}^3 \times \mathrm{SO}(3)$	$\mathrm{SO}(4)$	$\mathrm{SO}(3) \times \mathrm{SO}(3)$
$\mathrm{nil}\,\mathscr{H}(G)$	2	4	4	5

Theorem 1.2. *The groups* $\mathscr{H}(\mathrm{S}^3 \times \mathrm{SO}(3))$ *and* $\mathscr{H}(\mathrm{SO}(4))$ *are not isomorphic.*

Corollary 1.1. *The groups* $\mathrm{S}^3 \times \mathrm{SO}(3)$ *and* $\mathrm{SO}(4)$ *are of different H-homotopy type.*

We recall that $\mathrm{S}^3 \times \mathrm{SO}(3)$ and $\mathrm{SO}(4)$ are homeomorphic (cf. Section 2), though.

In Section 2, we calculate $\mathscr{H}(\mathrm{S}^3 \times \mathrm{SO}(3))$. In Section 3, we calculate $\mathscr{H}(\mathrm{SO}(4))$ and prove Theorem 1.2 and Corollary 1.1. In Section 4, we prove Theorem 1.1.

2. Notation and the group $\mathscr{H}(\mathrm{S}^3 \times \mathrm{SO}(3))$

Spaces are assumed to be based, maps and homotopies preserve base points, and the base point of G is the unit. We do not distinguish in notation between a map and its homotopy class. We denote the unit of a group by 0 or 1. The identity map of a space X is denoted by 1 or 1_X. Let

$$d_{n,X} : X \to X^{\wedge n} = X \wedge \cdots \wedge X \quad (n\text{-times})$$

be the reduced diagonal map. Let $C_G = C_{2,G} : G \wedge G \to G$ be the commutator map, that is, $C_G(x \wedge y) = [x, y] = xyx^{-1}y^{-1}$. The commutator product

$$[\ ,\ ] : [X, G] \times [X, G] \to [X, G]$$

is defined by $[f, g] = C_G \circ (f \wedge g) \circ d_{2,X}$, and the Samelson product

$$\langle\ ,\ \rangle : [X, G] \times [Y, G] \to [X \wedge Y, G]$$

is defined by $\langle f, g\rangle = C_G \circ (f \wedge g)$. We define $C_{n,G} : G^{\wedge n} \to G$ inductively by $C_{n,G} = C_G \circ (C_{n-1,G} \wedge 1)$ for $n \geqslant 3$. Given two spaces X_1, X_2, for $k = 1, 2$, let $\mathrm{pr}_k : X_1 \times X_2 \to X_k$ be the projection and $i_k : X_k \to X_1 \times X_2$ the inclusion. Let $\mathbb{Z}\{x\}$ be the infinite cyclic group generated by x and $\mathbb{Z}_n\{x\}$ the finite cyclic group of order n generated by x.

Let $\mathbf{P}^n$ be the real projective space of dimension n. In particular, $\mathrm{SO}(3) = \mathbf{P}^3$ and $\mathrm{SO}(2) = \mathrm{S}^1 = \mathbf{P}^1$. Let $p : \mathrm{S}^3 \to \mathbf{P}^3$ denote the double covering map. Then $p_* : \pi_k(\mathrm{S}^3) \cong \pi_k(\mathbf{P}^3)$ for $k \geqslant 2$. We abbreviate $C_{2,\mathbf{P}^3}$ and $C_{n,\mathbf{P}^3}$ by C and C_n, respectively. Let $\tilde{C} : \mathbf{P}^3 \wedge \mathbf{P}^3 \to \mathrm{S}^3$ be defined by $\tilde{C}(p(x) \wedge p(y)) = C_{\mathrm{S}^3}(x \wedge y)$. Then $\tilde{C} \circ (p \wedge p) = C_{\mathrm{S}^3}$ and $p \circ \tilde{C} = C$. Let $i_{k,n} : \mathbf{P}^k \to \mathbf{P}^n$ be the inclusion map ($k < n$) and $q_n : \mathbf{P}^n \to \mathbf{P}^n/\mathbf{P}^{n-1} = \mathrm{S}^n$ the quotient map. For $n \geqslant 3$, let $\eta_2 : \mathrm{S}^3 \to \mathrm{S}^2$ be the Hopf map and $\eta_n \in \pi_{n+1}(\mathrm{S}^n)$ the $(n-2)$-times suspension of η_2. Recall from [14] and p. 442 of [1] (cf. Section 9 of [13] and p. 176 of [4]) that

$$\pi_{n+1}(\mathrm{S}^n) = \mathbb{Z}_2\{\eta_n\} \quad (n \geqslant 3), \qquad \pi_{n+2}(\mathrm{S}^n) = \mathbb{Z}_2\{\eta_n^2\} \quad (n \geqslant 2),$$

$$\pi_6(\mathrm{S}^3) = \mathbb{Z}_{12}\{C_{\mathrm{S}^3}\}, \qquad \eta_3^3 = 6C_{\mathrm{S}^3},$$

where $\eta_n^2 = \eta_n \circ \eta_{n+1}$ and $\eta_n^3 = \eta_n \circ \eta_{n+1} \circ \eta_{n+2}$.

A compact, connected Lie group G of rank 2 is semi-simple and non-simple if and only if $G \cong (\mathrm{S}^3 \times \mathrm{S}^3)/H$ with H being a central subgroup. Since the center of $\mathrm{S}^3 \times \mathrm{S}^3$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ ([7,16]), such a Lie group G is isomorphic to one of the following:

$$\mathrm{S}^3 \times \mathrm{S}^3,\ \mathrm{SO}(3) \times \mathrm{S}^3,\ \mathrm{S}^3 \times \mathrm{SO}(3),\ (\mathrm{S}^3 \times \mathrm{S}^3)/d(\mathbb{Z}_2),\ \mathrm{SO}(3) \times \mathrm{SO}(3),$$

where $d : \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is the diagonal map. Recall from p. 99 of [16] and [7] that $\mathrm{SO}(4) = (\mathrm{S}^3 \times \mathrm{S}^3)/d(\mathbb{Z}_2)$ and note that $\mathrm{SO}(3) \times \mathrm{S}^3 \cong \mathrm{S}^3 \times \mathrm{SO}(3)$. Define

$$\varphi : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathrm{SO}(4)$$

by $\varphi(x, p(y)) = \pi(xy, y)$, where $\pi : \mathrm{S}^3 \times \mathrm{S}^3 \to \mathrm{SO}(4)$ is the projection. Then φ is a homeomorphism (but not an H-map, see Corollary 1.1). Note that

$$\mathscr{H}(\mathrm{S}^3 \times \mathrm{SO}(3)) \cong [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{S}^3] \oplus [\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3].$$

In the rest of this section, we shall calculate $[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{S}^3]$ and $[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3]$.

Lemma 2.1. (1) *For any group Γ and any elements x, y, z of it, we have*

$$[xy, z] = x[y, z]x^{-1}[x, z], \quad [x, yz] = [x, y]y[x, z]y^{-1}.$$

(2) *If $Y \cup e^n$ is a space with e^n, an n-cell, and $q : Y \cup e^n \to Y \cup e^n/Y = \mathrm{S}^n$ the quotient map, then the image of $q^* : \pi_n(G) \to [Y \cup e^n, G]$ is a central subgroup.*
(3) $[\mathbf{P}^3, \mathbf{P}^3] = \mathbb{Z}\{1_{\mathbf{P}^3}\}$ *and* $[\mathbf{P}^3, \mathrm{S}^3] = \mathbb{Z}\{q_3\}$.
(4) $\langle p, i_{1,3}\rangle = p \circ \eta_3$, $\langle\langle p, i_{1,3}\rangle, i_{1,3}\rangle = p \circ \eta_3^2$ *and* $\langle\langle\langle p, i_{1,3}\rangle, i_{1,3}\rangle, i_{1,3}\rangle = p \circ \eta_3^3 \neq 0$.
(5) $[\mathrm{S}^3 \wedge \mathbf{P}^2, \mathrm{S}^3] = \mathbb{Z}_4\{\tilde{C} \circ (p \wedge i_{2,3})\}$ *and* $(\tilde{C} \circ (p \wedge i_{2,3}))^2 = \eta_3^2 \circ (1_{\mathrm{S}^3} \wedge q_2)$.
(6) $[\mathrm{S}^3 \wedge \mathbf{P}^3, \mathrm{S}^3] = \mathbb{Z}_{12}\{\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3})\} \oplus \mathbb{Z}_4\{(C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3))^9\}$ *and*

$$(\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^8 = (C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3))^4,$$

$$C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3) = (\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^8 (C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3))^9.$$

(7) *Let $i_0 : A := (\mathrm{S}^3 \vee \mathbf{P}^3) \cup \mathrm{S}^3 \times \mathbf{P}^2 \to \mathrm{S}^3 \times \mathbf{P}^3$ be the inclusion map, and $q : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathrm{S}^3 \wedge \mathbf{P}^3$ and $\bar{q} : A \to A/(\mathrm{S}^3 \vee \mathbf{P}^3) = \mathrm{S}^3 \wedge \mathbf{P}^2$ the quotient maps. Then*

$$q \circ i_0 = (1_{\mathrm{S}^3} \wedge i_{2,3}) \circ \bar{q} \tag{2.1}$$

and the following sequence is exact:

$$0 \longrightarrow \pi_6(\mathbf{P}^3) \xrightarrow{q^*(1 \wedge q_3)^*} [\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3] \xrightarrow{i_0^*} [A, \mathbf{P}^3] \longrightarrow 0. \tag{2.2}$$

Proof. (1) is easy and well-known. As for (2) we refer to p. 91 of [3]. The first equality of (3) follows from Proposition 4.1 of [6] and the second equality is obvious.

Let

$$\mathrm{SO}(3) \xrightarrow{i'} \mathrm{SO}(4) \xrightarrow{p'} \mathrm{S}^3$$

be the canonical fibration. We prove $\langle p, i_{1,3}\rangle = p \circ \eta_3$ by showing $i' \circ \langle p, i_{1,3}\rangle \neq 0$. Let $j : \mathrm{U}(2) \to \mathrm{SO}(4)$ be the inclusion and $p'' : \mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(3)$ the projection.

Consider the following commutative diagram with exact horizontal sequences:

$$\begin{tikzcd}
\pi_4(\mathrm{SO}(5)/U(2)) \arrow[r] \arrow[d] & \pi_3(\mathrm{U}(2)) \arrow[r, "\cong"] \arrow[d, "j_*"] & \pi_3(\mathrm{SO}(5)) \arrow[r] \arrow[d, equal] & \pi_3(\mathrm{SO}(5)/U(2)) \arrow[d] \\
\pi_4(\mathrm{S}^4) \arrow[r, "\Delta"] \arrow[d, equal] & \pi_3(\mathrm{SO}(4)) \arrow[r] \arrow[d, "p'_*"] & \pi_3(\mathrm{SO}(5)) \arrow[r] \arrow[d, "p''_*"] & \pi_3(\mathrm{S}^4) \arrow[d, equal] \\
\pi_4(\mathrm{S}^4) \arrow[r] & \pi_3(\mathrm{S}^3) \arrow[r] & \pi_3(\mathrm{SO}(5)/\mathrm{SO}(3)) \arrow[r] & 0
\end{tikzcd}$$

Recall that

$$\pi_3(\mathrm{U}(2)) \cong \mathbb{Z}, \quad \pi_4(\mathrm{SO}(5)) \cong \mathbb{Z}_2, \quad \mathrm{SO}(5)/\mathrm{U}(2) \approx \mathrm{SO}(6)/\mathrm{U}(3) \text{ [2]},$$

$$\pi_k(\mathrm{SO}(6)/\mathrm{U}(3)) \cong \pi_{k+1}(\mathrm{SO}(\infty)) = 0 \quad (k = 3, 4).$$

Hence $\pi_3(\mathrm{SO}(4)) = \mathbb{Z}\{\Delta 1_{\mathrm{S}^4}\} \oplus \mathbb{Z}\{j_*\beta\}$ and $i' \circ p = a\Delta 1_{\mathrm{S}^4} + bj_*\beta$ with $a, b \in \mathbb{Z}$, where β generates $\pi_3(\mathrm{U}(2))$. Since the following exact sequence splits, $i' \circ p$ is not divisible by any integer $\geqslant 2$, and so a is prime to b:

$$0 \longrightarrow \pi_3(\mathrm{SO}(3)) \xrightarrow{i'_*} \pi_3(\mathrm{SO}(4)) \xrightarrow{p'_*} \pi_3(\mathrm{S}^3) \longrightarrow 0.$$

Let $p'_* j_* \beta = c \cdot 1_{\mathrm{S}^3}$, $c \in \mathbb{Z}$. Since the map p''_* in the diagram above is surjective and $\pi_3(\mathrm{SO}(5)/\mathrm{SO}(3)) \cong \mathbb{Z}_2$ [11], it follows that c is odd. Hence

$$0 = p'_* i'_* p = (\pm 2a + bc)1_{\mathrm{S}^3}$$

and so $bc = \mp 2a$. Thus, a is odd and b is even and so

$$i' \circ p \equiv \Delta 1_{\mathrm{S}^4} \pmod{2\pi_3(\mathrm{SO}(4))}$$

and

$$\begin{aligned}
i' \circ \langle p, i_{1,3}\rangle &= \langle i' \circ p, i' \circ i_{1,3}\rangle \quad \text{(by (15.2) and (15.3) of [5])} \\
&= \langle \Delta 1_{\mathrm{S}^4}, i' \circ i_{1,3}\rangle \quad (\text{since } \pi_4(\mathrm{SO}(4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2) \\
&= \pm\Delta\langle i' \circ i_{1,3}, 1_{\mathrm{S}^4}\rangle_{\mathrm{r}} \quad \text{(by (15.10) and (15.13) of [5])} \\
&= \pm\Delta J(i' \circ i_{1,3}) \quad \text{(by (16.2) of [5])},
\end{aligned}$$

where $\langle\, ,\rangle_{\mathrm{r}}$ is the relative Samelson product and $J : \pi_1(\mathrm{SO}(4)) \to \pi_5(\mathrm{S}^4)$ the J homomorphism [5]. Since $J(i' \circ i_{1,3}) = \eta_4$ and $\Delta\eta_4 \neq 0$, we have $i' \circ \langle p, i_{1,3}\rangle \neq 0$ as desired.

We have

$$\langle\langle p, i_{1,3}\rangle, i_{1,3}\rangle = \langle p \circ \eta_3, i_{1,3}\rangle = \langle p, i_{1,3}\rangle \circ \eta_4 = p \circ \eta_3^2,$$

where the second equality follows from (15.4) of [5]. Similarly,

$$\langle\langle\langle p, i_{1,3}\rangle, i_{1,3}\rangle, i_{1,3}\rangle = p \circ \eta_3^3,$$

which is not zero. This establishes (4).

We have

$$p \circ \tilde{C} \circ (p \wedge i_{2,3}) \circ (1_{\mathrm{S}^3} \wedge i_{1,2}) = C \circ (p \wedge i_{1,3}) = \langle p, i_{1,3}\rangle = p \circ \eta_3$$

and so $\tilde{C} \circ (p \wedge i_{2,3}) \circ (1_{S^3} \wedge i_{1,2}) = \eta_3$. Thus, (5) follows from Corollary 5(1) of [12] and the exactness of the sequence

$$0 \longrightarrow \pi_5(S^3) \xrightarrow{(1\wedge q_2)^*} [S^3 \wedge \mathbf{P}^2, S^3] \xrightarrow{(1\wedge i_{1,2})^*} \pi_4(S^3) \longrightarrow 0.$$

Since, as is well-known, the projection $S^2 \to \mathbf{P}^2$ is stably trivial, we have $S^3 \wedge \mathbf{P}^3 \simeq (S^3 \wedge \mathbf{P}^2) \vee S^6$. Hence

$$[S^3 \wedge \mathbf{P}^3, S^3] \cong [S^3 \wedge \mathbf{P}^2, S^3] \oplus \pi_6(S^3) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$$

and so the following exact sequence splits:

$$0 \longrightarrow \pi_6(S^3) \xrightarrow{(1\wedge q_3)^*} [S^3 \wedge \mathbf{P}^3, S^3] \xrightarrow{(1\wedge i_{2,3})^*} [S^3 \wedge \mathbf{P}^2, S^3] \longrightarrow 0.$$

Therefore $[S^3 \wedge \mathbf{P}^3, S^3]$ is generated by $C_{S^3} \circ (1_{S^3} \wedge q_3)$ and $\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3})$. Since $(1_{S^3} \wedge p)^*(\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3})) = C_{S^3}$, it follows that the order of $\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3})$ is 12 and that $(1_{S^3} \wedge p)^* : [S^3 \wedge \mathbf{P}^3, S^3] \to \pi_6(S^3)$ is an epimorphism. Since

$$(1_{S^3} \wedge p)^*\{(C_{S^3} \circ (1_{S^3} \wedge q_3))(\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^{-2}\} = 1,$$

the order of $(C_{S^3} \circ (1_{S^3} \wedge q_3))(\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^{-2}$ is 4. Hence

$$(C_{S^3} \circ (1_{S^3} \wedge q_3))^4 = (\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^8$$

and so

$$\begin{aligned} C_{S^3} \circ (1_{S^3} \wedge q_3) &= (C_{S^3} \circ (1_{S^3} \wedge q_3))^4 (C_{S^3} \circ (1_{S^3} \wedge q_3))^9 \\ &= (\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}))^8 (C_{S^3} \circ (1_{S^3} \wedge q_3))^9. \end{aligned}$$

Thus,

$$[S^3 \wedge \mathbf{P}^3, S^3] = \mathbb{Z}_{12}\{\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3})\} \oplus \mathbb{Z}_4\{(C_{S^3} \circ (1_{S^3} \wedge q_3))^9\}.$$

This proves (6).

The identity (2.1) is obvious from the definitions. In the following commutative diagram, two vertical sequences and the first horizontal sequence are short exact:

$$\begin{tikzcd}
\pi_6(\mathbf{P}^3) \arrow[r, "(1\wedge q_3)^*"] \arrow[d, equal] & {[S^3 \wedge \mathbf{P}^3, \mathbf{P}^3]} \arrow[r] \arrow[d, "q^*"] & {[S^3 \wedge \mathbf{P}^2, \mathbf{P}^3]} \arrow[d, "\bar{q}^*"] \\
\pi_6(\mathbf{P}^3) \arrow[r, "q^*(1\wedge q_3)^*"] & {[S^3 \times \mathbf{P}^3, \mathbf{P}^3]} \arrow[r, "i_0^*"] \arrow[d] & {[A, \mathbf{P}^3]} \arrow[d] \\
 & {[S^3 \vee \mathbf{P}^3, \mathbf{P}^3]} \arrow[r, equal] & {[S^3 \vee \mathbf{P}^3, \mathbf{P}^3].}
\end{tikzcd}$$

Hence the second horizontal sequence is also short exact. This establishes (7) and completes the proof of Lemma 2.1. □

Proposition 2.1. *We have* $[S^3 \times \mathbf{P}^3, S^3] = H \oplus \mathbb{Z}_4\{(\tilde{C} \circ (p \wedge 1_{\mathbf{P}^3}) \circ q)^9\}$, *where* H *is the image of the monomorphism* $(1 \times q_3)^* : [S^3 \times S^3, S^3] \to [S^3 \times \mathbf{P}^3, S^3]$. *The group* $[S^3 \times S^3, S^3]$ *is generated by* $\mathrm{pr}_1, \mathrm{pr}_2$ *and* $C_{S^3} \circ \tilde{q}$, *subject to the relations* $(C_{S^3} \circ \tilde{q})^{12} = 1$ *and* $[\mathrm{pr}_1, \mathrm{pr}_2] = C_{S^3} \circ \tilde{q}$, *where* $\tilde{q} : S^3 \times S^3 \to S^3 \wedge S^3$ *is the quotient map.*

Proof. For the second assertion, cf. Proposition 3.1 of [6]. Since

$$[\mathrm{pr}_1, q_3 \circ \mathrm{pr}_2] = C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q$$

in $[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{S}^3]$ and since the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_6(\mathrm{S}^3) & \xrightarrow{\tilde{q}^*} & [\mathrm{S}^3 \times \mathrm{S}^3, \mathrm{S}^3] & \longrightarrow & [\mathrm{S}^3 \vee \mathrm{S}^3, \mathrm{S}^3] & \longrightarrow & 0 \\
 & & \cap \Big\downarrow (1\wedge q_3)^* & & \Big\downarrow (1\times q_3)^* & & \cong \Big\downarrow (1\vee q_3)^* & & \\
0 & \longrightarrow & [\mathrm{S}^3 \wedge \mathbf{P}^3, \mathrm{S}^3] & \xrightarrow{q^*} & [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{S}^3] & \longrightarrow & [\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3] & \longrightarrow & 0,
\end{array}$$

the first assertion is a consequence of Lemma 2.1. □

Proposition 2.2. *The group* $[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3]$ *is generated by*

$$z_1 = p \circ \mathrm{pr}_1, \qquad z_2 = \mathrm{pr}_2, \qquad z_3 = C \circ (p \wedge 1_{\mathbf{P}^3}) \circ q, \qquad z_4 = \tilde{z}_4^9$$

subject to the relations

$$z_3^{12} = z_4^4 = [z_1, z_3] = [z_i, z_4] = 1 \quad (1 \leqslant i \leqslant 3),$$

$$[z_1, z_2] = z_3, \qquad [z_2, z_3] = z_3^6 z_4, \tag{2.3}$$

where $\tilde{z}_4 = p \circ C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q$.

Proof. Since $p_* : [\mathrm{S}^3 \wedge \mathbf{P}^3, \mathrm{S}^3] \cong [\mathrm{S}^3 \wedge \mathbf{P}^3, \mathbf{P}^3]$, it follows from Lemma 2.1(6) and the exactness of the following sequence that the group $[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3]$ is generated by z_1, z_2, z_3, and z_4:

$$0 \longrightarrow [\mathrm{S}^3 \wedge \mathbf{P}^3, \mathbf{P}^3] \xrightarrow{q^*} [\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3] \longrightarrow [\mathrm{S}^3 \vee \mathbf{P}^3, \mathbf{P}^3] \longrightarrow 0.$$

We have $[z_1, z_3] = 1$ and $[z_1, z_2] = z_3$ from the definition of commutator products. Moreover, Lemma 2.1(2) implies that z_4 is central so that $[z_i, z_4] = 1$ for $1 \leqslant i \leqslant 3$. By Lemma 2.1(6), we have

$$z_3^{12} = z_4^4 = 1, \qquad z_3^8 = \tilde{z}_4^4, \qquad \tilde{z}_4 = z_3^8 z_4. \tag{2.4}$$

The cellular approximation theorem implies that there is a map

$$f : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathrm{S}^3 \wedge \mathbf{P}^2 \wedge \mathbf{P}^2$$

such that

$$(q \wedge \mathrm{pr}_2) \circ d_{2,\mathrm{S}^3 \times \mathbf{P}^3} = (1_{\mathrm{S}^3} \wedge i_{2,3} \wedge i_{2,3}) \circ f.$$

We have

$$\begin{aligned}
[z_3, z_2] &= C_3 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 2}) \circ (q \wedge \mathrm{pr}_2) \circ d_{2,\mathrm{S}^3 \times \mathbf{P}^3} \\
&= C \circ (p \wedge 1_{\mathbf{P}^3}) \circ (\tilde{C} \wedge 1_{\mathbf{P}^3}) \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 2}) \circ (q \wedge \mathrm{pr}_2) \circ d_{2,\mathrm{S}^3 \times \mathbf{P}^3} \\
&= C \circ (p \wedge 1_{\mathbf{P}^3}) \circ \{(\tilde{C} \circ (p \wedge i_{2,3})) \wedge i_{2,3}\} \circ f.
\end{aligned}$$

Since Lemma 2.1(5) implies that the order of $\tilde{C} \circ (p \wedge i_{2,3})$ is 4, we have

$$[z_3, z_2]^4 = 1. \tag{2.5}$$

The cellular approximation theorem implies that there is a map

$$g : \mathrm{S}^5 \to \mathrm{S}^3 \wedge \mathbf{P}^{1\wedge 2} = \mathrm{S}^5$$

such that

$$(\bar{q} \wedge \mathrm{pr}_2|_A) \circ d_{2,A} = (1_{\mathrm{S}^3} \wedge i_{1,2} \wedge i_{1,3}) \circ g \circ (1_{\mathrm{S}^3} \wedge q_2) \circ \bar{q}.$$

Applying $H^5(-; \mathbb{Z}_2)$ to the last equality, we see that the degree of g is odd. We have

$$\begin{aligned} i_0^*[z_3, z_2] &= [i_0^* z_3, i_0^* z_2] \\ &= C_3 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 2}) \circ (1_{\mathrm{S}^3} \wedge i_{2,3} \wedge 1_{\mathbf{P}^3}) \circ (\bar{q} \wedge \mathrm{pr}_2|_A) \circ d_{2,A} \\ &= C_3 \circ (p \wedge i_{1,3}^{\wedge 2}) \circ g \circ (1_{\mathrm{S}^3} \wedge q_2) \circ \bar{q} = p \circ \eta_3^2 \circ g \circ (1_{\mathrm{S}^3} \wedge q_2) \circ \bar{q} \\ &= p \circ \eta_3^2 \circ (1_{\mathrm{S}^3} \wedge q_2) \circ \bar{q} \\ &= \{C \circ (p \wedge i_{2,3}) \circ \bar{q}\}^2 \quad \text{(by Lemma 2.1(5))} \\ &= \{C \circ (p \wedge 1_{\mathbf{P}^3}) \circ q \circ i_0\}^2 \quad \text{(by (2.1))} \\ &= i_0^* z_3^2 \end{aligned}$$

and so

$$[z_3, z_2] z_3^{-2} \in \mathrm{Ker}(i_0^*) = \mathrm{Im}\{q^*(1_{\mathrm{S}^3} \wedge q_3)^*\} = \mathbb{Z}_{12}\{\tilde{z}_4\}$$

by (2.2). Hence $[z_3, z_2] z_3^{-2} = \tilde{z}_4^a$ for some integer a. Thus,

$$[z_3, z_2] = z_3^2 \tilde{z}_4^a$$

and so $z_3^{8(a+1)} = 1$ by (2.4) and (2.5). Therefore, $a \equiv 2 \pmod 3$. We shall show $a \equiv 5 \pmod 6$ by proving

$$[[[z_1, z_2], z_2], z_2] = p \circ \eta_3^3 \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q \neq 1. \tag{2.6}$$

Assume (2.6). Then, writing $a = 3a' + 2$, we have

$$\begin{aligned} [[[z_1, z_2], z_2], z_2] &= [[z_3, z_2], z_2] = [z_3^2 \tilde{z}_4^a, z_2] = [z_3^2, z_2] \\ &= z_3 [z_3, z_2] z_3^{-1} [z_3, z_2] = z_3^4 \tilde{z}_4^{2a} = z_3^4 \tilde{z}_4^4 \tilde{z}_4^{6a'} = \tilde{z}_4^{6a'}. \end{aligned}$$

Hence a' is odd and so $a \equiv 5 \pmod 6$. Thus, $\tilde{z}_4^{6a'} = \tilde{z}_4^6 = \tilde{z}_4^{18} = z_4^2$ and

$$[[[z_1, z_2], z_2], z_2] = z_4^2.$$

Furthermore, by (2.4), we have

$$[z_3, z_2] = z_3^2 \tilde{z}_4^a = z_3^{2+8a} z_4^a = z_3^6 z_4^{\varepsilon}$$

with $\varepsilon = \pm 1$ and so $[z_2, z_3] = z_3^6 z_4^{-\varepsilon}$. We have $\tilde{z}_4(x, y) = [p(x), (p \circ q_3)(y)]$ for $(x, y) \in \mathrm{S}^3 \times \mathbf{P}^3$, and so, since $p \circ q_3 = 1_{\mathbf{P}^3}^2 : \mathbf{P}^3 \to \mathbf{P}^3$, we have

$$\begin{aligned}\tilde{z}_4 &= [p \circ \mathrm{pr}_1, (\mathrm{pr}_2)^2] = [z_1, z_2^2] = [z_1, z_2] z_2 [z_1, z_2] z_2^{-1}\\ &= z_3 z_2 z_3 z_2^{-1} = z_3 (z_3^6 z_4^{-\varepsilon} z_3 z_2) z_2^{-1} = z_3^8 z_4^{-\varepsilon} = \tilde{z}_4^{4-9\varepsilon}.\end{aligned}$$

Since the order of $\tilde{z}_4$ is 12, it follows that $\varepsilon = -1$ and so $[z_2, z_3] = z_3^6 z_4$. This settles (2.3) and we have

$$\tilde{z}_4 = [z_1, z_2^2] = z_3^8 z_4. \tag{2.7}$$

Now we prove (2.6). From the definition of commutator products, we have

$$[[[z_1, z_2], z_2], z_2] = C_4 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 3}) \circ k,$$

where $k = (\mathrm{pr}_1 \wedge \mathrm{pr}_2^{\wedge 3}) \circ d_{4, \mathrm{S}^3 \times \mathbf{P}^3}$. The cellular approximation theorem implies that there is a map

$$h : \mathrm{S}^6 \to \mathrm{S}^3 \wedge \mathbf{P}^{1\wedge 3} = \mathrm{S}^6$$

such that

$$k = (1_{\mathrm{S}^3} \wedge i_{1,3}^{\wedge 3}) \circ h \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q.$$

Applying $H^6(-; \mathbb{Z}_2)$ to the last equality, we see that the degree of h is odd. Then

$$\begin{aligned}[[[z_1, z_2], z_2], z_2] &= C_4 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 3}) \circ (1_{\mathrm{S}^3} \wedge i_{1,3}^{\wedge 3}) \circ h \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q\\ &= p \circ \eta_3^3 \circ h \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q \quad \text{(by Lemma 2.1(4))}\\ &= p \circ \eta_3^3 \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q\\ &\neq 1 \quad \text{(by (2.2))}.\end{aligned}$$

This completes the proof of Proposition 2.2. □

Let $\Lambda = [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{S}^3]$ and $\Gamma = [\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3]$. Then $\mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3) \cong \Lambda \oplus \Gamma$ and, by Propositions 2.1 and 2.2, we have

$$[\Lambda, \Lambda] = \mathbb{Z}_{12}\{C_{\mathrm{S}^3} \circ (1_{\mathrm{S}^3} \wedge q_3) \circ q\}, \qquad [\Gamma, \Gamma] = \mathbb{Z}_{12}\{z_3\} \oplus \mathbb{Z}_4\{z_4\},$$

$$[[\Gamma, \Gamma], \Gamma] = \mathbb{Z}_4\{z_3^6 z_4\}, \qquad [[[\Gamma, \Gamma], \Gamma], \Gamma] = \mathbb{Z}_2\{z_4^2\}.$$

Hence $\mathrm{nil}\,\Lambda = 2$ and $\mathrm{nil}\,\Gamma = 4$.

Corollary 2.1. *The group* $\mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ *is nilpotent of class* 4 *and its commutator subgroup is isomorphic to* $\mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_4$.

3. The group $\mathscr{H}(\mathrm{SO}(4))$

Since $\varphi : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathrm{SO}(4)$ is the homeomorphism, it induces the isomorphism

$$\varphi^* : \mathscr{H}(\mathrm{SO}(4)) \cong [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)]. \tag{3.1}$$

Hence it suffices to compute $[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)]$. We have the following commutative diagram with short exact rows:

$$\begin{tikzcd}
{[\mathrm{S}^3 \wedge \mathbf{P}^3, \mathrm{SO}(4)]} \arrow[r] \arrow[d, "\pi_*", "\cong"'] & {[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)]} \arrow[r] \arrow[d, "\pi_*", "\cap"'] & {[\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{SO}(4)]} \arrow[d, "\pi_*", "\cap"'] \\
{[\mathrm{S}^3 \wedge \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3]} \arrow[r, "q^*"] & {[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3]} \arrow[r] & {[\mathrm{S}^3 \vee \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3].}
\end{tikzcd} \tag{3.2}$$

Lemma 3.1. *The image of the composite*

$$[\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{SO}(4)] \xrightarrow{\pi_*} [\mathrm{S}^3 \vee \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3] \xrightarrow{\psi} \pi_3(\mathbf{P}^3 \times \mathbf{P}^3) \oplus \mathscr{H}(\mathbf{P}^3) \oplus \mathscr{H}(\mathbf{P}^3)$$

is $\pi_3(\mathbf{P}^3 \times \mathbf{P}^3) \oplus \{(2\alpha + \beta, \beta) \mid \alpha, \beta \in \mathscr{H}(\mathbf{P}^3)\}$ *where* ψ *is the canonical isomorphism.*

Proof. The image of $\psi \circ \pi_*$ is the direct sum of the images of the following two homomorphisms:

$$\pi_* : \pi_3(\mathrm{SO}(4)) \to \pi_3(\mathbf{P}^3 \times \mathbf{P}^3), \tag{3.3}$$

$$\pi_* : [\mathbf{P}^3, \mathrm{SO}(4)] \to [\mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3] = \mathscr{H}(\mathbf{P}^3) \oplus \mathscr{H}(\mathbf{P}^3). \tag{3.4}$$

The map π_* in (3.3) is an isomorphism. Hence it suffices to show that the image of π_* in (3.4) is $\{(2\alpha + \beta, \beta) \mid \alpha, \beta \in \mathscr{H}(\mathbf{P}^3)\}$. Since

$$\varphi_* : [\mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3] \to [\mathbf{P}^3, \mathrm{SO}(4)]$$

is a bijection, Lemma 2.1(3) implies that any element of the image of π_* in (3.4) has the form $\pi \circ \varphi \circ \alpha$, where

$$\alpha = (i_1 \circ q_3^l)(i_2 \circ 1_{\mathbf{P}^3}^m) \in [\mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3]$$

for some $l, m \in \mathbb{Z}$. We have

$$\pi \circ \varphi \circ \alpha(p(x)) = ((p \circ q_3 \circ p(x))^l p(x)^m, p(x)^m).$$

Hence $\mathrm{pr}_2 \circ \pi \circ \varphi \circ \alpha = 1_{\mathbf{P}^3}^m$ and $\mathrm{pr}_1 \circ \pi \circ \varphi \circ \alpha = 1_{\mathbf{P}^3}^{2l+m}$. Therefore, the image of π_* in (3.4) is $\{(2\alpha + \beta, \beta) \mid \alpha, \beta \in \mathscr{H}(\mathbf{P}^3)\}$. $\square$

Maintaining the notation of Proposition 2.2, we define certain elements of $[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3]$ as follows: $x_k = i_1 \circ z_k$, $y_k = i_2 \circ z_k$ $(1 \leqslant k \leqslant 4)$. Let K denote the image of the induced map

$$\pi_* : [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)] \to [\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3].$$

The following is a consequence of Lemma 2.1(1), Proposition 2.2, (2.7), (3.2), and Lemma 3.1.

Theorem 3.1. *The map* π_* *yields an isomorphism from* $[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)]$ *onto* K. *The group* K *is generated by* $x_1,\ y_1,\ x_2^2,\ x_2y_2,\ x_3,\ y_3,\ x_4,\ y_4$, *subject to the relations*

$$x_3^{12} = y_3^{12} = x_4^4 = y_4^4 = 1,$$

$$[x_1, x_3] = [y_1, y_3] = [y_1, x_2^2] = [x_2^2, x_2y_2] = [x_2^2, x_3] = [x_2^2, y_3] = 1,$$

$$[x_k, y_l] = 1 \quad (\text{for all } k, l \text{ with } k, l \neq 2),$$

$$[\alpha, x_4] = 1 \quad (\text{for all } \alpha \in \{x_1, x_2^2, x_2y_2, x_3\}),$$

$$[\alpha, y_4] = 1 \quad (\text{for all } \alpha \in \{y_1, y_2, x_2^2, x_2y_2, y_3\}),$$

$$[x_1, x_2^2] = x_3^8x_4, \qquad [x_1, x_2y_2] = x_3, \qquad [y_1, x_2y_2] = y_3,$$

$$[x_2y_2, x_3] = x_3^6x_4, \qquad [x_2y_2, y_3] = y_3^6y_4.$$

Hence the commutator subgroup of K *amounts to*

$$\mathbb{Z}_{12}\{x_3\} \oplus \mathbb{Z}_{12}\{y_3\} \oplus \mathbb{Z}_4\{x_4\} \oplus \mathbb{Z}_4\{y_4\}.$$

Since

$$\mathrm{nil}\,\mathscr{H}(\mathrm{SO}(4)) = \mathrm{nil}\,[\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)] \leqslant \mathrm{nil}[\mathrm{S}^3 \times \mathbf{P}^3, \mathbf{P}^3 \times \mathbf{P}^3] = 4$$

and $[[[x_1, x_2y_2], x_2y_2], x_2y_2] = x_4^2 \neq 0$, we obtain

Corollary 3.1. *The nilpotency class of* $\mathscr{H}(\mathrm{SO}(4))$ *is* 4.

Proof of Theorem 1.2. Since Corollary 2.1 and Theorem 3.1 imply that the commutator subgroups of $\mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ and $\mathscr{H}(\mathrm{SO}(4))$ have distinct orders, it follows that $\mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ and $\mathscr{H}(\mathrm{SO}(4))$ are not isomorphic. □

Proof of Corollary 1.1. If there were an H-map $f : \mathrm{SO}(4) \to \mathrm{S}^3 \times \mathbf{P}^3$ which is a homotopy equivalence, then $f_* : [\mathrm{S}^3 \times \mathbf{P}^3, \mathrm{SO}(4)] \to \mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ is an isomorphism and so we have an isomorphism $\mathscr{H}(\mathrm{SO}(4)) \cong \mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ by (3.1), this would contradict Theorem 1.2. □

4. The proof of Theorem 1.1

We have $\mathrm{nil}\,\mathscr{H}(\mathrm{S}^3 \times \mathrm{S}^3) = 2$ by Proposition 2.1 and

$$\mathrm{nil}\,\mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3) = \mathrm{nil}\,\mathscr{H}(\mathrm{SO}(4)) = 4$$

by Corollaries 2.1 and 3.1. In order to establish Theorem 1.1, it suffices to prove the following.

Proposition 4.1. *The group* $\mathscr{H}(\mathbf{P}^3 \times \mathbf{P}^3)$ *has nilpotency class* 5.

We need a lemma. Let $[\mathrm{S}^3] \in H^3(\mathrm{S}^3;\mathbb{Z}_2)$ and $t \in H^1(\mathbf{P}^3;\mathbb{Z}_2)$ be generators of respective groups.

Lemma 4.1. *The class* $\tilde{C}^*[\mathrm{S}^3]$ *may be written as* $\tilde{C}^*[\mathrm{S}^3] = t^2 \otimes t + t \otimes t^2$.

Proof. Since $q_2^* : H^2(\mathrm{S}^2;\mathbb{Z}_2) \cong H^2(\mathbf{P}^2;\mathbb{Z}_2)$ and since the composite of

$$H^3(\mathrm{S}^3;\mathbb{Z}_2) \xrightarrow{\tilde{C}^*} H^3(\mathbf{P}^3 \wedge \mathbf{P}^3;\mathbb{Z}_2) \cong H^3(\mathbf{P}^2 \wedge \mathbf{P}^1;\mathbb{Z}_2) \oplus H^3(\mathbf{P}^1 \wedge \mathbf{P}^2;\mathbb{Z}_2)$$

is $\{\tilde{C} \circ (i_{2,3} \wedge i_{1,3})\}^* \oplus \{\tilde{C} \circ (i_{1,3} \wedge i_{2,3})\}^*$, it suffices to prove the following equalities:

$$\tilde{C} \circ (i_{2,3} \wedge i_{1,3}) = q_2 \wedge 1_{\mathbf{P}^1}, \qquad \tilde{C} \circ (i_{1,3} \wedge i_{2,3}) = 1_{\mathbf{P}^1} \wedge q_2.$$

As is easily seen, $[\mathbf{P}^2 \wedge \mathbf{P}^1, \mathrm{S}^3] = \mathbb{Z}_2\{q_2 \wedge 1_{\mathbf{P}^1}\}$ and $[\mathbf{P}^1 \wedge \mathbf{P}^2, \mathrm{S}^3] = \mathbb{Z}_2\{1_{\mathbf{P}^1} \wedge q_2\}$. To induce a contradiction, assume $\tilde{C} \circ (i_{2,3} \wedge i_{1,3}) = 0$. Then

$$\tilde{C} \circ (1_{\mathbf{P}^3} \wedge i_{1,3}) = f \circ (q_3 \wedge 1_{\mathbf{P}^1})$$

for some $f \in \pi_4(\mathrm{S}^3)$. Since $f \circ 2 \cdot 1_{\mathrm{S}^4} = 0$ and $q_3 \circ p = 2 \cdot 1_{\mathrm{S}^3}$, we have

$$0 = p \circ f \circ (q_3 \wedge 1_{\mathbf{P}^1}) \circ (p \wedge 1_{\mathbf{P}^1}) = C \circ (1_{\mathbf{P}^3} \wedge i_{1,3}) \circ (p \wedge 1_{\mathbf{P}^1}) = \langle p, i_{1,3} \rangle.$$

This contradicts Lemma 2.1(4). Hence $\tilde{C} \circ (i_{2,3} \wedge i_{1,3}) \neq 0$, that is,

$$\tilde{C} \circ (i_{2,3} \wedge i_{1,3}) = q_2 \wedge 1_{\mathbf{P}^1}.$$

Similarly, $\tilde{C} \circ (i_{1,3} \wedge i_{2,3}) = 1_{\mathbf{P}^1} \wedge q_2$. □

Proof of Proposition 4.1. Since

$$\mathscr{H}(\mathbf{P}^3 \times \mathbf{P}^3) \cong [\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] \oplus [\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3],$$

it suffices to prove $\mathrm{nil}[\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] = 5$. First, we prove $\mathrm{nil}[\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] \geqslant 5$ by showing

$$\delta := [[[[\mathrm{pr}_1, \mathrm{pr}_2], \mathrm{pr}_2], \mathrm{pr}_2], \mathrm{pr}_1] \neq 0.$$

Let $q' : \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbf{P}^3 \wedge \mathbf{P}^3$ be the quotient map. We have $[\mathrm{pr}_1, \mathrm{pr}_2] = p \circ \tilde{C} \circ q'$ and hence, from the definition, $\delta = C_4 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 3}) \circ k$, where

$$k = (\tilde{C} \wedge 1_{\mathbf{P}^3}^{\wedge 3}) \circ (q' \wedge \mathrm{pr}_2 \wedge \mathrm{pr}_2 \wedge \mathrm{pr}_1) \circ d_{4, \mathbf{P}^3 \times \mathbf{P}^3}.$$

The cellular approximation theorem implies that there is a map

$$g : \mathrm{S}^6 \to \mathrm{S}^6 = \mathrm{S}^3 \wedge \mathbf{P}^{1\wedge 3}$$

such that

$$k = (1_{\mathrm{S}^3} \wedge i_{1,3}^{\wedge 3}) \circ g \circ (q_3 \wedge q_3) \circ q'.$$

Apply $H^6(-;\mathbb{Z}_2)$ to the last equality. Then Lemma 4.1 implies that

$$t^3 \otimes t^3 = k^*([\mathrm{S}^3] \otimes t \otimes t \otimes t) = q'^*(q_3 \wedge q_3)^* g^*[\mathrm{S}^6],$$

where $[\mathrm{S}^6]$ is the generator of $H^6(\mathrm{S}^6;\mathbb{Z}_2)$. Hence $g^*[\mathrm{S}^6] \neq 0$ so that the degree of g is odd and hence

$$\delta = C_4 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 3}) \circ (1_{\mathrm{S}^3} \wedge i_{1,3}^{\wedge 3}) \circ g \circ (q_3 \wedge q_3) \circ q'$$

$$= p \circ \eta_3^3 \circ g \circ (q_3 \wedge q_3) \circ q' = p \circ \eta_3^3 \circ (q_3 \wedge q_3) \circ q'.$$

As was shown in p. 300 of [12], $(q_3 \wedge q_3)^* : \pi_6(\mathbf{P}^3) \to [\mathbf{P}^3 \wedge \mathbf{P}^3, \mathbf{P}^3]$ is injective. Hence ${q'}^*(q_3 \wedge q_3)^* : \pi_6(\mathbf{P}^3) \to [\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3]$ is a monomorphism. Therefore $\delta \neq 0$, because $p \circ \eta_3^3 \neq 0$.

Next we prove $\mathrm{nil}[\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] \leqslant 5$ by showing

$$\varepsilon := [[[[[u_1, u_2], u_3], u_4], u_5], u_6] = 0$$

for all $u_1, \ldots, u_6 \in [\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3]$. Since $[u_1, u_2] \in {q'}^*[\mathbf{P}^3 \wedge \mathbf{P}^3, \mathbf{P}^3]$ and

$$p_* : [\mathbf{P}^3 \wedge \mathbf{P}^3, \mathrm{S}^3] \cong [\mathbf{P}^3 \wedge \mathbf{P}^3, \mathbf{P}^3],$$

we can write $[u_1, u_2] = p \circ f \circ q'$ for some $f \in [\mathbf{P}^3 \wedge \mathbf{P}^3, \mathrm{S}^3]$. Then $\varepsilon = h \circ g$, where

$$g = ((f \circ q') \wedge u_3 \wedge u_4 \wedge u_5 \wedge u_6) \circ d_{5, \mathbf{P}^3 \times \mathbf{P}^3} : \mathbf{P}^3 \times \mathbf{P}^3 \to \mathrm{S}^3 \wedge {\mathbf{P}^3}^{\wedge 4},$$

$$h = C_5 \circ (p \wedge 1_{\mathbf{P}^3}^{\wedge 4}) : \mathrm{S}^3 \wedge {\mathbf{P}^3}^{\wedge 4} \to \mathbf{P}^3.$$

We have $g = 0$, because $\mathrm{S}^3 \wedge {\mathbf{P}^3}^{\wedge 4}$ is six-connected and $\dim \mathbf{P}^3 \times \mathbf{P}^3 = 6$. Hence $\varepsilon = 0$ and so $\mathrm{nil}[\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] \leqslant 5$.

Therefore, $\mathrm{nil}[\mathbf{P}^3 \times \mathbf{P}^3, \mathbf{P}^3] = 5$. $\square$

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References

[1] A. Borel, J. Serre, Groupes de Lie et puissances réduites de Steenrod, Amer. J. Math. 75 (1953) 409–448.
[2] B. Harris, Some calculations of homotopy groups of symmetric spaces, Trans. Amer. Math. Soc. 106 (1963) 174–184.
[3] P. Hilton, G. Mislin, J. Roitberg, Localization of Nilpotent Groups and Spaces, in: Mathematics Studies, Vol. 15, North-Holland/American Elsevier, Amsterdam, 1975.
[4] I. James, On H-spaces and their homotopy groups, Quart. J. Math. Oxford 11 (1960) 161–179.
[5] I. James, The Topology of Stiefel Manifolds, in: London Mathematical Society Lecture Notes, Vol. 24, Cambridge University Press, Cambridge, 1976.
[6] M. Mimura, H. Ōshima, Self homotopy groups of Hopf spaces with at most three cells, J. Math. Soc. Japan 51 (1999) 71–92.
[7] M. Mimura, H. Toda, Topology of Lie Groups I, in: Translations of Mathematical Monographs, Vol. 91, American Mathematical Society, Rhode Island, Providence, 1991.
[8] H. Ōshima, Self homotopy set of a Hopf space, Quart. J. Math. Oxford 50 (1999) 483–495.
[9] H. Ōshima, Self homotopy group of the exceptional Lie group G_2, J. Math. Kyoto Univ. 40 (2000) 177–184.

[10] H. Ōshima, N. Yagita, Non commutativity of self homotopy groups, Kodai Math. J. 24 (2001) 15–25.
[11] G. Paechter, The groups $\pi_r(V_{n,m})$ (I), Quart. J. Math. Oxford 7 (1956) 249–268.
[12] E. Rees, Multiplications on projective spaces, Michigan Math. J. 19 (1968) 297–301.
[13] H. Samelson, Groups and spaces of loops, Comment. Math. Helv. 28 (1954) 178–287.
[14] H. Toda, Composition Methods in Homotopy Groups of Spheres, in: Annals of Mathematical Studies, Vol. 49, Princeton University Press, Princeton, 1962.
[15] G. Whitehead, On mappings into group-like spaces, Comment. Math. Helv. 28 (1954) 320–328.
[16] I. Yokota, Classical Simple Lie Groups, Gendai Sugaku Sha, Kyoto, 1990 (in Japanese).
[17] I. Yokota, Exceptional Simple Lie Groups, Gendai Sugaku Sha, Kyoto, 1992 (in Japanese).